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Local and Global Duality and the Determination of $\alpha(M_Z)$

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Abstract. This talk[1] presents work concepts and results for the determination of the fine structure constant α at the Z_0 mass resonance. The problem consisting of the break-down of global duality for singular integral weights is circumvented by using a polynomial fit which mimics this weight function. This method is conservative in the sense that it is mostly independent of special assumptions. In this context the difference between local and global duality is explained.

INTRODUCTION

There is a great deal of interest in the accurate determination of the running fine structure constant α at the scale of the Z_0 mass [1–4]. The value of $\alpha(M_Z)$ is of paramount importance for all precision tests of the Standard Model. Furthermore, an accurate knowledge of $\alpha(M_Z)$ is instrumental in narrowing down the the mass window for the last missing particle of the Standard Model, the Higgs particle.

The main source of uncertainty in the determination of $\alpha(M_Z)$ is the hadronic contribution to the e^+e^- annihilations needed for this evaluation. The necessary dispersion integral that enters this calculation has in the past been evaluated by using experimental e^+e^- annihilation data. Discrepancies in the experimental data between different experiments suggest large systematic uncertainties in each of the experiments. In order to reduce the influence of the systematic uncertainties on the determination of $\alpha(M_Z)$ one may attempt to add some theoretical input to the evaluation of the hadronic contribution to $\alpha(M_Z)$.

M. Davier and A. Höcker [3] use QCD perturbation theory in form of local duality (explained later on) in the region above $s = (1.8\,\mathrm{GeV})^2$ for the light flavours, while J.H. Kühn and M. Steinhauser [5] use perturbative results for energy regions outside the charm and bottom threshold regions. Our approach [6] is quite different. We attempt to minimize the influence of data in the dispersion integral over the whole energy region including the threshold regions.

[1)] Talk given at the conference "High Energy Physics at the Millennium (MRST'99)", Ottawa, Canada, May 10–12, 1999

Local and global duality

For the e^+e^- annihilation process there is a main connection between the spectral density $\rho(s)$ and the two-point correlator $\Pi(q^2)$ given in kind of the dispersion relation

$$\Pi(q^2) = \int_{s_0}^{\infty} \frac{\rho(s)ds}{s+q^2} \tag{1}$$

which implies the reverse relation

$$\rho(s) = \frac{1}{2\pi i} Disc\, \Pi(s) \tag{2}$$

where the discontinuity is given by

$$Disc\, \Pi(q^2) = \Pi(q^2 e^{-i\pi}) - \Pi(q^2 e^{i\pi}) \tag{3}$$

and $s_0 = 4m_\pi^2$ is the production threshold of the light flavours. These relations can be a chain between the theory, i.e. the two-point correlator function within perturbative QCD on the one hand and the experiment, i.e. the spectral density or, equivalently, the total cross section on the other hand. But there is one obstacle in using these relations: As depending on methods of functional analysis, the inverse of the dispersion relation is only valid if there are no poles encircled by the path in the complex plane, a condition which is necessary to obtain this relation. These poles can have their origin from weight functions in combination with the spectral density. This means that if there is such a weight function included in the integration of the spectral density, the inverse relation shown above is only valid *locally* and not *globally*. We call this *local* resp. *global duality*.

The experiment side

The hadronic contribution to $\alpha(M_Z)$ which we are concentrating on is given by the integral [1]

$$\Delta\alpha_{\rm had}(M_Z) = \frac{\alpha}{3\pi} Re \int_{s_0}^{\infty} R(s)H(s)ds \tag{4}$$

where $R(s)$ is the total e^+e^- hadronic cross section and $H(s)$ is the weight function

$$H(s) = \frac{M_Z^2}{s(M_Z^2 - s)}. \tag{5}$$

The hadronic cross section is related to the spectral density by

$$R(s) = 12\pi^2 \rho(s). \tag{6}$$

But we see: the weight function $H(s)$ is indeed singular at the points $s = 0$ and $s = M_Z^2$ on the real axis. So global duality is not valid any more.

The theory side

The two-point correlator is given by

$$i\int\langle 0|j_\alpha^{\rm em}(x)j_\beta^{\rm em}(0)|0\rangle e^{iqx}d^4x=(-g_{\alpha\beta}q^2+q_\alpha q_\beta)\Pi(q^2) \tag{7}$$

where we only included the isospin contribution $I=1$, in contrast to corresponding considerations for the τ decay. The scalar correlator function $\Pi(q^2)$ consists of perturbative and non-perturbative contributions which we include to the extend we need them to keep the accuracy. For the perturbative contribution to the correlator we use a result given in ref. [7]. I only write down the first few terms,

$$\Pi^{\rm P}(q^2)=\frac{3}{16\pi^2}\sum_{i=1}^{n_f}Q_i^2\left[\frac{20}{9}+\frac{4}{3}L+C_F\left(\frac{55}{12}-4\zeta(3)+L\right)\frac{\alpha_s}{\pi}+O(\alpha_s^2,m_q^2/q^2)\right] \tag{8}$$

with $L=\ln(\mu^2/q^2)$ while in ref. [7] the expression is given up to $O(\alpha_s^2,m_q^{12}/q^{12})$. The number of active flavours is denoted by n_f. For the zeroth order term in the m_q^2/q^2 expansion we have added higher order terms in α_s,

$$\frac{3}{16\pi^2}\sum_{i=1}^{n_f}Q_i^2\left[\left(c_3+3k_2L+\frac{1}{2}(k_0\beta_1+2k_1\beta_0)L^2\right)\left(\frac{\alpha_s}{\pi}\right)^3+O(\alpha_s^4)\right] \tag{9}$$

with $k_0=1$, $k_1=1.63982$ and $k_2=6.37101$. We have denoted the yet unknown constant term in the four-loop contribution by c_3. Remark, however, that the constant non-logarithmic terms will not contribute to our calculations. The non-perturbative contributions are given in ref. [8],

$$\begin{aligned}\Pi^{\rm NP}(q^2)=&\frac{1}{18q^4}\left(1+\frac{7\alpha_s}{6\pi}\right)\langle\frac{\alpha_s}{\pi}G^2\rangle\\&+\frac{8}{9q^4}\left(1+\frac{\alpha_s}{4\pi}C_F+\dots\right)\langle m_u\bar uu\rangle+\frac{2}{9q^4}\left(1+\frac{\alpha_s}{4\pi}C_F+\dots\right)\langle m_d\bar dd\rangle\\&+\frac{2}{9q^4}\left(1+\frac{\alpha_s}{4\pi}C_F+(5.8+0.92L)\frac{\alpha_s^2}{\pi^2}\right)\langle m_s\bar ss\rangle\\&+\frac{\alpha_s^2}{9\pi^2q^4}(0.6+0.333L)\langle m_u\bar uu+m_d\bar dd\rangle\\&-\frac{C_Am_s^4}{36\pi^2q^4}\left(1+2L+(0.7+7.333L+4L^2)\frac{\alpha_s}{\pi}\right)\\&-\frac{448\pi}{243q^6}\alpha_s|\langle\bar qq\rangle|^2+O(q^{-8})\end{aligned} \tag{10}$$

where we have included the m_s^4/q^4-contribution arising from the unit operator. In this expression we used the $SU(3)$ colour factors $C_F=4/3$, $C_A=3$, and $T_F=1/2$. For the coupling constant α_s as well as for the running quark mass we use four-loop expression given in refs. [9–11] even though in both cases the three-loop accuracy would already have been sufficient for the present application.

INTRODUCING OUR METHOD

Our method is based on the fact that we can use global duality when the weight function is non-singular. This is the case for a polynomial function. So we mimic the weight function by a polynomial function obeying different conditions which we will explain later. By adding and subtracting this polynomial function $P_N(s)$ of given order N to the weight function $H(s)$, we obtain without any restrictions

$$\int_{s_a}^{s_b} \rho(s)H(s)ds = \int_{s_a}^{s_b} \rho(s)\left(H(s) - P_N(s)\right)ds + \int_{s_a}^{s_b} \rho(s)P_N(s)ds \tag{11}$$

where $[s_a, s_b]$ is any interval out of the total integration range. But because the second term has now a polynomial weight, we can use global duality to write

$$\begin{aligned}\int_{s_a}^{s_b} \rho(s)P_N(s)ds &= \frac{1}{2\pi i}\int_{s_a}^{s_b} \mathit{Disc}\,\Pi(s)P_N(s)ds \;= \\ &= -\frac{1}{2\pi i}\oint_{|s|=s_a} \Pi(-s)P_N(s)ds + \frac{1}{2\pi i}\oint_{|s|=s_b} \Pi(-s)P_N(s)ds. \end{aligned}\tag{12}$$

Thus this part can be represented by a difference of two circle integrals in the complex plane. On the other hand, the difference $H(s) - P_N(s)$ suppresses the contribution of the first part. Our method consists thus of the following steps:

- replacing $\rho(s)$ in the first part of Eq. (11) by the value of the experimentally measured total cross section $R(s)$ (see e.g. ref. [1])
- replacing the circle integral contribution to flavours at their threshold by zero
- in all other cases inserting the QCD perturbative and non-perturbative parts of $\Pi(-s)$ on the circle

These replacements can be seen as a concept within QCD sum rules. To obtain the best efficience of our method, we have to restrict the polynomial function by the following contraints:

- The method of least squares should be used to mimic the weight
- However, the degree N should not be higher than the order of the highest perturbative resp. non-perturbative contribution increased by one (this is a consequence of the Cauchy's theorem which is involved in the analytical integration of the circle integrals)
- Especially for the low energy region, the polynomial function should vanish on the real axis to avoid instanton effects
- In regions where resonances occur, the polynomial function should fit the weight function to suppress those contributions which constitute the highest uncertainty of the experimental data

As just mentioned, the integration on the circle can be done analytically by using the Cauchy's theorem. But we have to keep in mind that the result for $\Pi(-s)$ we use here depends logarithmically on the renormalization scale μ and on the parameters of the theory that are renormalized at the scale μ. These are the strong coupling constant, the quark masses and the condensates. As advocated in [12], we implement the renormalization group improvement for the moments of the electromagnetic correlator by performing the integrations over the circle with radius $|s| = s_b$ with constant parameters, i.e. they are renormalized at a fixed scale μ. Subsequently these parameters are evolved from this scale to $\mu^2 = s_a$ using the four-loop β function. In other words, we impose the renormalization group equation on the moments rather than on the correlator itself. This procedure is not only technically simpler but also avoids possible inconsistencies inherent to the usual approach where one applies the renormalization group to the correlator, expands in powers of $\ln(s/\mu^2)$ and carries out the integration in the complex plane only at the end. In the present case the reference scale is given by $\Lambda_{\overline{\rm MS}}$.

Subdividing the integration interval

As a first interval we select the range from the light flavour production threshold $s_0 = 4m_\pi^2$ and the next threshold marked by the mass of the ψ, $s_1 = m_\psi^2 \approx (3.1\,\mathrm{GeV})^2$. In this case we set the inner circle integral to zero and obtain

$$\int_{s_0}^{s_1} R(s)H(s)ds = \int_{s_0}^{s_1} R^{\rm exp}(s)\,(H(s) - P_N(s))\,ds + 6\pi i \oint_{|s|=s_1} \Pi^{\rm QCD}(-s)P_N(s)ds. \tag{13}$$

As mentioned above, we impose the constraints to the polynomial function that it should vanish on the real axis at $s = s_1$ and should coincide with the weight function at the ρ resonance, i.e. for $s = m_\rho^2 \approx (1\,\mathrm{GeV})^2$. Fig. 1 shows polynomials of different order in comparison with the weight function. The results shown in Fig. 2 are compared with the result obtained by using only the experimental data. For the up and down quarks we only keep the mass independent part of the QCD contribution while for the strange quark we include also the terms to order $O(m_s^2/q^2)$.

The second interval is limited by s_1 and the threshold marked by the mass of the Υ, $s_2 = m_\Upsilon^2 \approx (9.46\,\mathrm{GeV})^2$. For the charm quark, we again set the inner circle integral to zero, but for the lighter quarks we have to keep both. The perturbative series for the charm quark is used up to it's known extend.

The third interval given between s_2 and $(40\,\mathrm{GeV})^2$ is again subdivided into two pieces because of it's length. For the first of these two intervals we choose $[(9.46\,\mathrm{GeV})^2, (30\,\mathrm{GeV})^2]$, for the second $[(30\,\mathrm{GeV})^2, (40\,\mathrm{GeV})^2]$. Now the bottom quark is the one for which the "threshold rule" (i.e. leaving out the inner circle) applies. The remaining part of the integral starting from $s_4 = (40\,\mathrm{GeV})^2$ up to infinity is done using local duality, i.e. by inserting the function $R(s)$ obtained for perturbative QCD into the second part of Eq. (11).

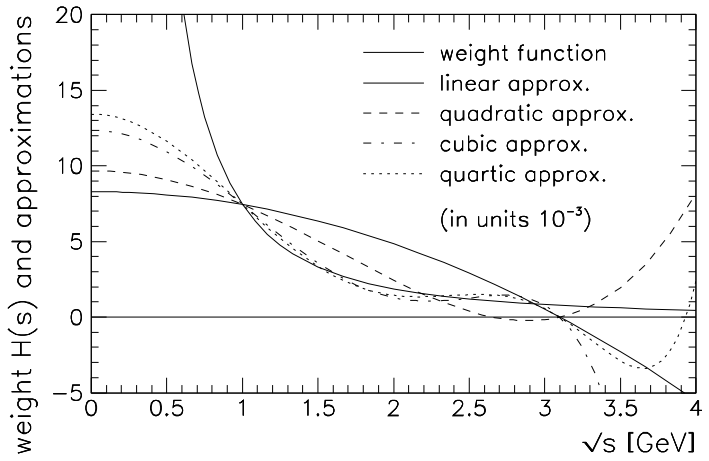

FIGURE 1. Weight function $H(s)$ and polynomial approximations $P_N(s)$ in the lowest energy interval $2m_\pi \leq \sqrt{s} \leq 3.1\,\mathrm{GeV}$. The least square fit was done in the interval $m_\rho \leq \sqrt{s} \leq 3.1\,\mathrm{GeV}$ with further constraints $H(s) = P_N(s)$ at $\sqrt{s} = 1\,\mathrm{GeV}$ and $P_N(s) = 0$ at $\sqrt{s} = 3.1\,\mathrm{GeV}$. The quality of the polynomial approximations are shown up to $N = 4$. We use the scaled variable s/s_1 for the polynomial approximation where s_1 is the upper radius such that $P_N(s/s_1)$ is dimensionless.

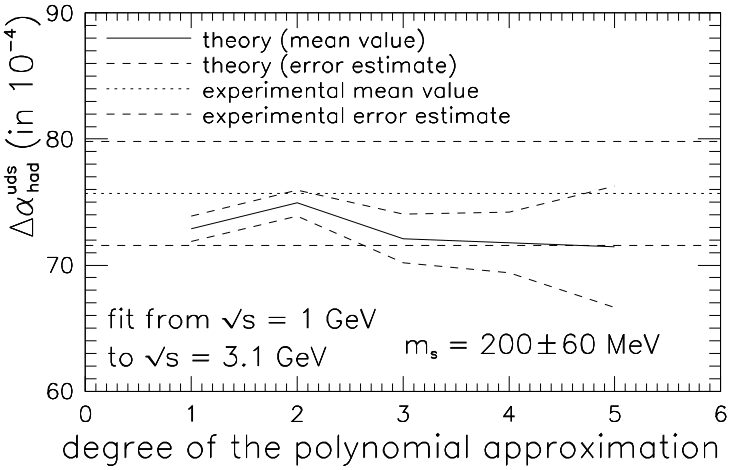

FIGURE 2. Comparison of the l.h.s. and r.h.s. of the sum rule given by Eq. (13) in the interval $0.28\,\mathrm{GeV} \leq \sqrt{s} \leq 3.1\,\mathrm{GeV}$. Dotted horizontal line: value of integrating the l.h.s. using experimental data including error bars [1]. The points give the values of the r.h.s. integration for various orders N of the polynomial approximation. Straight line interpolations between the points are for illustration only. The dashed lines indicate the error estimate of our calculation.

Our results are collected in Table 1. To obtain these results, we used the condensate values

$$\langle\frac{\alpha_s}{\pi}GG\rangle = (0.04 \pm 0.04)\,\mathrm{GeV}^4, \qquad \alpha_s\langle\bar{q}q\rangle^2 = (4 \pm 4)\cdot 10^{-4}\,\mathrm{GeV}^6. \tag{14}$$

For the errors coming from the uncertainty of the QCD scale we take

$$\Lambda_{\overline{\mathrm{MS}}} = 380 \pm 60\,\mathrm{MeV} \tag{15}$$

The errors resulting from the uncertainty in the QCD scale in different energy intervals are clearly correlated and will have to be added linearly in the end. We also include the error of the strange quark mass in the light quark region which is taken as

$$\bar{m}_s(1\,\mathrm{GeV}) = 200 \pm 60\,\mathrm{MeV} \tag{16}$$

For the charm and bottom quark masses we use the values

$$\bar{m}_c(m_c) = 1.4 \pm 0.2\,\mathrm{GeV}, \quad \bar{m}_b(m_b) = 4.8 \pm 0.3\,\mathrm{GeV}. \tag{17}$$

Summing up the contributions from the five flavours u, d, s, c, and b, our result for the hadronic contribution to the dispersion integral including the systematic error due to the dependence on $\Lambda_{\overline{\mathrm{MS}}}$ (column 5 in Table 1) reads

$$\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z) = (277.6 \pm 4.1)\cdot 10^{-4}. \tag{18}$$

In order to obtain the total result for $\alpha(M_Z)$, we have to add the lepton and top contributions. Since we have nothing new to add to the calculation of these contributions we simply take the values from ref. [5],

$$\Delta\alpha^{t}_{\mathrm{had}}(M_Z) = (-0.70 \pm 0.05)\cdot 10^{-4}, \qquad \Delta\alpha_{\mathrm{lep}}(M_Z) \approx 314.97\cdot 10^{-4}. \tag{19}$$

Writing $\Delta\alpha(M_Z) = \Delta\alpha_{\mathrm{lep}}(M_Z) + \Delta\alpha_{\mathrm{had}}(M_Z)$ our final result is ($\alpha(0)^{-1} = 137.036$)

$$\alpha(M_Z)^{-1} = \alpha(0)^{-1}(1 - \Delta\alpha(M_Z)) = 128.925 \pm 0.056. \tag{20}$$

CONCLUSION AND OUTLOOK

I have presented a method to obtain the running fine structure constant α at the scale of the Z_0 mass with minimal input of experimental data. This method is conservative in the meaning that its error is as free from assumptions as possible. Our method is discussed and compared with other methods (see e.g. ref. [13] – however, with our preliminary results). In ref. [14] Matthias Steinhauser says that "it is very impressive that the new analysis show very good agreement both in their central values and their quoted errors." I nevertheless would like to close this talk with the remark that all recent calculations of $\alpha(M_Z)$ should not deter experimentalists from remeasuring the e^+e^- annihilation cross section more accurately in the low and intermediate energy region, as such data are absolutely essential for a precise value of $\alpha(M_Z)$, unbiased by theory.

TABLE 1. Contributions of different energy intervals to $\alpha^{(5)}_{\rm had}(M_Z)$. Second column: choice of neighbouring pairs of the polynomial degree N. Third column: fraction of the contribution of experimental data [1]. Fourth column: contribution to $\Delta\alpha^{(5)}_{\rm had}(M_Z)$ with all errors included except for the systematic error due to the dependence on $\Lambda_{\overline{\rm MS}}$ which is separately listed in the fifth column.

interval for $\sqrt{s}$	values of N	data contribution	contribution to $\Delta\alpha^{(5)}_{\rm had}(M_Z)$	error due to $\Lambda_{\overline{\rm MS}}$
$[0.28\,\text{GeV}, 3.1\,\text{GeV}]$	$1, 2$	24%	$(73.9 \pm 1.1) \cdot 10^{-4}$	$0.9 \cdot 10^{-4}$
$[3.1\,\text{GeV}, 9.46\,\text{GeV}]$	$3, 4$	0.3%	$(69.5 \pm 3.0) \cdot 10^{-4}$	$1.4 \cdot 10^{-4}$
$[9.46\,\text{GeV}, 30\,\text{GeV}]$	$3, 4$	1.1%	$(71.6 \pm 0.5) \cdot 10^{-4}$	$0.06 \cdot 10^{-4}$
$[30\,\text{GeV}, 40\,\text{GeV}]$	$3, 4$	0.15%	$(19.93 \pm 0.01) \cdot 10^{-4}$	$0.02 \cdot 10^{-4}$
$\sqrt{s} > 40\,\text{GeV}$			$(42.67 \pm 0.09) \cdot 10^{-4}$	
total range			$(277.6 \pm 3.2) \cdot 10^{-4}$	$1.67 \cdot 10^{-4}$

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REFERENCES

1. S. Eidelman and F. Jegerlehner, *Z. Phys.* **C67**, 585 (1995)
2. H. Burkhardt and B. Pietrzyk, *Phys. Lett.* **356 B**, 398 (1995)
3. M. Davier and A. Höcker, *Phys. Lett.* **419 B**, 419 (1998)
4. M.L. Swartz, *Phys. Rev.* **D53**, 5268 (1996)
5. J.H. Kühn and M. Steinhauser, *Phys. Lett.* **437 B**, 425 (1998)
6. S. Groote, J.G. Körner, K. Schilcher and N.F. Nasrallah, *Phys. Lett.* **440 B**, 375 (1998)
7. K.G. Chetyrkin, R. Harlander, J.H. Kühn and M. Steinhauser, *Nucl. Phys.* **B503**, 339 (1997)
8. S.G. Gorishny, A.L. Kataev and S.A. Larin, *Nuovo Cimento* **92 A**, 119 (1986)
9. K.G. Chetyrkin, B.A. Kniehl and M. Steinhauser, *Phys. Rev. Lett.* **79**, 353 (1997); *ibid.*, Report No. MPI/PhT/97-041, hep-ph/9708255
10. N. Gray, D.J. Broadhurst, W. Grafe and K. Schilcher, *Z. Phys.* **C48**, 673 (1990)
11. K.G. Chetyrkin, *Phys. Lett.* **404 B**, 161 (1997)
12. K.G. Chetyrkin, D. Pirjol and K. Schilcher, *Phys. Lett.* **404 B**, 337 (1997)
13. M. Davier and A. Höcker, *Phys. Lett.* **435 B**, 427 (1998)
14. M. Steinhauser, "Hadronic contribution to $\alpha(M_Z^2)$ and the anomalous magnetic moment of the muon", talk given at the 17$^{\rm th}$ International Workshop on Weak Interactions and Neutrinos (WIN 99), Cape Town, South Africa, January 24–30, 1999, Report No. BUTP-99/06, hep-ph/9904373